HW 2 Problem 2

Scheffe's lemma: Suppose $f_n \to f$ a.e. and $\int |f| d\mu < \infty$. Show that

$$\int f_n \to \int f \, d\mu \iff \int |f_n - f| \to 0$$

Pf:

Note the following improvement of DOM:

Let $|f_n| \leq g_n$ $\quad g_n \in \mathcal{L}^1$, $f_n \to f$ a.s.

If $\underline{\lim} \int g_n - \int \underline{\lim}\, g_n \to 0$, then $\int f_n \to \int f$.

If $\int \underline{\lim}\, g_n = +\infty$, then $\underline{\lim} \int g_n - \int \underline{\lim}\, g_n$ is not defined. Thus $\Rightarrow$ $\underline{\lim} \int g_n < \infty$, and so

Fatou $\Rightarrow$ $\int |f| \leq \underline{\lim} \int g_n < \infty$, and $\underline{\lim} \int |f_n| < +\infty$ by Monotonicity.

Note $\quad g_n + f_n \geq 0$, $\quad g_n - f_n \geq 0$

By Fatou:

$\Rightarrow$ $\underline{\lim} \int g_n + \underline{\lim} \int f_n \geq \int \underline{\lim}\, g_n + \int f$

$\Rightarrow$ $\underline{\lim} \int g_n + \underline{\lim} \int -f_n \geq \int \underline{\lim}\, g_n - \int f$

Since $\int f$ is integrable, we can use

$\Rightarrow$ $\overline{\lim} \int f_n \leq \int f \leq \underline{\lim} \int f_n$

This gives $\lim \int f_n = \int f$

Return to our problem, and note if $\int |f_n - f| \longrightarrow 0$ then

$$\int |f_n| \leq \int |f| + \int |f_n - f| \quad \text{by MON} \Rightarrow f_n \in L^1$$

Thus $$\left| \int f_n - \int f \right| \leq \int |f_n - f| \longrightarrow 0 \quad \Rightarrow \quad \int f_n \to \int f$$

For the other direction: If $\int f_n \longrightarrow \int f$

Write $$\int |f_n - f| = \int (f_n - f) 1_{\{f_n > f\}} + \int (f_n - f) 1_{\{f_n < f\}}$$

Since $f_n, f \geqslant 0 \Rightarrow$

$$0 \leq (f_n - f) 1_{\{f_n > f\}} \leqslant f_n + f \quad g_n$$

We are in the setup of DOM, and $\underline{\lim} \int g_n = \underline{\lim} \int f_n + f = 2 \int f = \int \underline{\lim}\, g_n$

$= 2 \int f$. (By assumption)

Therefore $(f_n - f) 1_{\{f_n > f\}} \longrightarrow 0$

$\Rightarrow \int (f_n - f) 1_{\{f_n > f\}}$ and vice-versa